

# Duality Theorem and Drinfeld Double in Braided Tensor Categories \*

Shouchuan Zhang

Department of Mathematics, Hunan University, Changsha 410082, P.R.China  
E-mail: z9491@yahoo.com.cn

## Abstract

Let  $H$  be a finite Hopf algebra with  $C_{H,H} = C_{H,H}^{-1}$ . The duality theorem is shown for  $H$ , i.e.,

$$(R \# H) \# H^* \cong R \otimes (H \bar{\otimes} H^*) \quad \text{as algebras in } \mathcal{C}.$$

Also, it is proved that the Drinfeld double  $(D(H), [b])$  is a quasi-triangular Hopf algebra in  $\mathcal{C}$ .

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## 1 Introduction and Preliminaries

It is well-known that in the work of  $C^*$ -algebras and von Neumann algebras, for an abelian group  $G$ , the product of  $R$  crossed by  $G$  crossed by  $\hat{G}$  is isomorphic to the tensor product of  $R$  and the compact operator. Its generalization to Hopf-von Neumann algebras was known again. (see, for example, Stratila [13]). Blattner and Montgomery strip off the functional analysis and duplicate the result at the level of Hopf algebras (see [2] and [11]). They proved that for an ordinary Hopf algebra  $H$  and some subalgebra  $U$  of  $H^*$ ,

$$(R \# H) \# U \cong R \otimes (H \# U) \quad \text{as algebras}.$$

The basic construction of the Drinfeld double is due to Drinfeld [4]. S.Majid [7] and D.E.Radford [12] modified the treatment.

In this paper, we generalize the duality theorem and Drinfeld double into the braided case, i.e., for a finite Hopf algebra  $H$  with  $C_{H,H} = C_{H,H}^{-1}$ , we show that

$$(R \# H) \# H^* \cong R \otimes (H \bar{\otimes} H^*) \quad \text{as algebras in } \mathcal{C}$$

We also show that the Drinfeld double  $(D(H), [b])$  is a quasi-triangular Hopf algebra in  $\mathcal{C}$ .

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In this paper,  $(\mathcal{C}, \otimes, I, C)$  is always a braided tensor category, where  $I$  is the identity object and  $C$  is the braiding.

By [16, Theorem 0.1], we can view every braided tensor category as a strict braided tensor category.

For an object  $V$  in  $\mathcal{C}$ , if there exists an object  $U$  and morphisms  $d_V : U \otimes V \rightarrow I$  and  $b_V : I \rightarrow V \otimes U$  in  $\mathcal{C}$  such that  $(d_V \otimes id_U)(id_U \otimes b_V) = id_U$  and  $(id_V \otimes d_V)(b_V \otimes id_V) = id_V$ , then  $U$  is called a left dual of  $V$ , written as  $V^*$ . In this case,  $d_V$  and  $b_V$  are called the evaluation morphism and coevaluation morphism of  $V$ , respectively. In general, we use  $d$  and  $b$  instead of  $d_V$  and  $b_V$ . Furthermore,  $V$  is said to be finite if  $V$  has a left dual (see [15]).

Let us define the transpose  $f^* = (d \otimes id_{U^*})(id_{V^*} \otimes f \otimes id_{U^*})(id_{V^*} \otimes b) : V^* \rightarrow U^*$  of a morphism  $f : U \rightarrow V$ .

Let  $\Delta^{cop} = C_{H,H}\Delta$  and  $m^{op} = mC_{H,H}$ . We denote  $(H, \Delta^{cop}, \epsilon)$  by  $H^{cop}$  and  $(H, m^{op}, \eta)$  by  $H^{op}$ . Furthermore we denote  $((H^*)^{op})^{cop}$  by  $H^*$ .

A bialgebra  $(H, R, \bar{\Delta})$  with convolution-invertible  $R$  in  $Hom_{\mathcal{C}}(I, H \otimes H)$  is called a quasi-triangular bialgebra in  $\mathcal{C}$  if  $(H, \bar{\Delta}, \epsilon)$  is a coalgebra and the following conditions are satisfied:

- (QT1)  $(\bar{\Delta} \otimes id)R = (id \otimes id \otimes m)(id \otimes R \otimes id)R$ ;
- (QT2)  $(id \otimes \Delta)R = (m \otimes id \otimes id)(id \otimes R \otimes id)R$ ;
- (QT3)  $(m \otimes m)(id \otimes C_{H,H} \otimes id)(\bar{\Delta} \otimes R) = (m \otimes m)(id \otimes C_{H,H} \otimes id)(R \otimes \Delta)$ .

In this case, we also say that  $(H, R, \bar{\Delta})$  is a braided quasi-triangular bialgebra.

Dually, we can define a coquasi-triangular bialgebra  $(H, r, \bar{m})$  in the braided tensor category  $\mathcal{C}$ .

In particular, we say that  $(H, R)$  is quasi-triangular if  $(H, R, \Delta^{cop})$  is quasi-triangular. Dually, we say that  $(H, r)$  is coquasi-triangular if  $(H, r, m^{op})$  is coquasi-triangular.

A morphism  $\tau$  from  $H \otimes A$  to  $I$  in  $\mathcal{C}$  is called a skew pairing on  $H \otimes A$  if the following conditions are satisfied:

- (SP1):  $\tau(m \otimes id) = \tau(id \otimes \tau \otimes id)(id \otimes id \otimes \Delta)$ ;
- (SP2):  $\tau(id \otimes m) = (\tau \otimes \tau)(id \otimes C_{H,A} \otimes id)(\Delta \otimes id \otimes id)$ ;
- (SP3):  $\tau(id \otimes \eta) = \epsilon_H$ ;
- (SP4):  $\tau(\eta \otimes id) = \epsilon_A$ .

If a morphism  $\tau$  from  $V \otimes W$  to  $I$  in  $\mathcal{C}$  satisfies

$$(id_U \otimes \tau)(C_{V,U} \otimes id_W) = (\tau \otimes id_U)(id_V \otimes C_{U,W})$$

for  $U = V, W$ , then  $\tau$  is said to be symmetric with respect to the braiding  $C$ .

**Lemma 1.1** *If  $H$  has a left dual  $H^*$ , then the following conditions are equivalent:*

- (i) *The evaluation of  $H$  is symmetric with respect to the braiding  $C$ .*
- (ii)  *$C_{U,V} = C_{V,U}^{-1}$  for  $U, V = H$  or  $H^*$ .*
- (iii)  *$C_{H,H} = C_{H,H}^{-1}$ .*
- (iv)  *$C_{H^*,H^*} = C_{H^*,H^*}^{-1}$ .*
- (v)  *$(id_H \otimes d)(C_{H^*,H} \otimes id_H) = (d \otimes id_H)(id_{H^*} \otimes C_{H,H})$ .*
- (vi)  *$(id_{H^*} \otimes d)(C_{H^*,H^*} \otimes id_H) = (d \otimes id_{H^*})(id_{H^*} \otimes C_{H^*,H})$ .*

**Proof.** It is straightforward.  $\square$

## 2 The Duality Theorem

In this section, we obtain the duality theorem for Hopf algebras living in the braided tensor category  $\mathcal{C}$ .

Throughout this section,  $H$  is a finite Hopf algebra with  $C_{H,H} = C_{H,H}^{-1}$  living in  $\mathcal{C}$ .  $(R, \alpha)$  is a left  $H$ -module algebra in  $\mathcal{C}$  and  $R \# H$  is the smash product in  $\mathcal{C}$ . Let  $H^{\hat{*}} = ((H^*)^{op})^{cop}$ .

The proof of Lemmas 2.1 — 2.7 is very similar to that of corresponding results in [11].

**Lemma 2.1** *If  $(R, \alpha)$  is an  $H$ -module algebra, let  $\phi = (id \otimes \alpha)(b' \otimes id) : R \rightarrow H^* {}^{op} \otimes R$ , where  $b' = C_{H,H^*} {}^{op} b$ . Then*

$$\phi m = (m \otimes m)(id \otimes C_{R,H^* {}^{op}} \otimes id)(\phi \otimes \phi).$$

**Lemma 2.2** (1)  $(H, \rightharpoonup)$  is a left  $H^{\hat{*}}$ -module algebra under the module operation  $\rightharpoonup = (id \otimes d)(C_{H^*,H} \otimes id)(id \otimes \Delta)$ .

(2)  $(H^*, \rightharpoonup)$  is a left  $H$ -module algebra under the module operation  $\rightharpoonup = (id \otimes d)(id \otimes C_{H,H^*})(C_{H,H^*} \otimes id)(id \otimes \Delta)$ .

(3)  $(H, \leftharpoonup)$  is a right  $H^{\hat{*}}$ -module algebra under the module operation  $\leftharpoonup = (d \otimes id)(C_{H,H^*} \otimes id)(id \otimes C_{H,H^*})(\Delta \otimes id)$ .

(4)  $(H^*, \leftharpoonup)$  is a right  $H$ -module algebra under the module operation  $\leftharpoonup = (d \otimes id)(id \otimes C_{H^*,H})(\Delta \otimes id)$ .

**Lemma 2.3** *The object  $H \otimes H^{\hat{*}}$  becomes an algebra, written as  $H \bar{\otimes} H^{\hat{*}}$ , under the multiplication  $m_{H \bar{\otimes} H^{\hat{*}}} = (id_H \otimes d \otimes id_{H^{\hat{*}}})$  and unity  $\eta_{H \bar{\otimes} H^{\hat{*}}} = b$ .*

In fact, if  $\mathcal{C}$  is a braided tensor category determined by the (co)quasi-triangular structure of a (co)quasi-triangular Hopf algebra over a field  $k$ , then  $H \bar{\otimes} H^{\hat{*}}$  can be viewed as  $End_k(H)$  or  $M_n(k) = \{A \mid A \text{ is an } n \times n \text{ matrix over the field } k\}$ .

**Lemma 2.4** *Let  $\lambda = (m \otimes d \otimes id)(id \otimes C_{H^*,H} \otimes id \otimes id)(id \otimes id \otimes \Delta \otimes id)(id \otimes id \otimes b)$  and  $\rho = (d \otimes m \otimes id)(id \otimes id \otimes C_{H,H} \otimes id)(id \otimes C_{H,H} \otimes id \otimes id)(id \otimes id \otimes \Delta \otimes id)(id \otimes id \otimes b)$ . Then  $\lambda$  is an algebra morphism from  $H \# H^{\hat{*}}$  to  $H \bar{\otimes} H^{\hat{*}}$  and  $\rho$  is an anti-algebra morphism from  $H^{\hat{*}} \# H$  to  $H \bar{\otimes} H^{\hat{*}}$ .*

**Lemma 2.5** *The following relation holds:  $m(\lambda \otimes \rho) = m(\rho \otimes \lambda)(id \otimes \rightharpoonup \otimes \leftharpoonup \otimes id)(id \otimes C_{H,H^*} \otimes C_{H^*,H} \otimes id)(id \otimes id \otimes C_{H^*,H^*} \otimes id \otimes id)(id \otimes C_{H^*,H} \otimes C_{H,H^*} \otimes id)(C_{H^*,H^*} \otimes id \otimes id \otimes C_{H^*,H^*})(S \otimes id \otimes id \otimes id \otimes id)(\Delta \otimes id \otimes id \otimes \Delta)(id \otimes C_{H,H} \otimes id)(C_{H,H^*} \otimes C_{H^*,H})(id \otimes C_{H^*,H^*} \otimes id)$ .*

**Lemma 2.6** *If the antipode of  $H$  is invertible, then  $\lambda$  is invertible.*

**Lemma 2.7**  $R\#H$  becomes an  $H^*$ -module algebra under the module operation  $\rightharpoonup' = (id \otimes \rightharpoonup)(C_{H^*,R} \otimes id)$ .

**Theorem 2.8** If  $H$  is a finite Hopf algebra with  $C_{H,H} = C_{H,H}^{-1}$ , then

$$(R\#H)\#H^{\hat{*}} \cong R \otimes (H \bar{\otimes} H^{\hat{*}}) \quad \text{as algebras in } \mathcal{C},$$

Where  $H \bar{\otimes} H^{\hat{*}}$  is defined in Lemma 2.3.

**Proof.** We first define a morphism  $w$  from  $H^{\hat{*}}$  to  $H\#H^{\hat{*}}$  such that  $w = \lambda^{-1}\rho(S^{-1} \otimes \eta_H)$ ; this can be done by Lemma 2.6 and [15, Theorem 4.1]. Since  $\rho$  and  $S^{-1}$  are anti-algebra morphisms,  $w$  is an algebra morphism. Set  $\phi = (id \otimes \alpha)(C_{H,H^{\hat{*}}} \otimes id)(b \otimes id)$ .

We now define a morphism  $\Phi = (id \otimes m_{H\#H^{\hat{*}}})(id \otimes w \otimes id \otimes id)(C_{H^*,R} \otimes id \otimes id)(\phi \otimes id \otimes id)$  from  $(R\#H)\#H^{\hat{*}}$  to  $R \otimes (H \bar{\otimes} H^{\hat{*}})$  and a morphism  $\Psi = (id \otimes m_{H\#H^{\hat{*}}})(id \otimes w \otimes id \otimes id)(C_{H^*,R} \otimes id \otimes id)(S \otimes id \otimes id \otimes id)(\phi \otimes id \otimes id)$  from  $R \otimes (H \bar{\otimes} H^{\hat{*}})$  to  $(R\#H)\#H^{\hat{*}}$ . It is straightforward to verify that  $\Phi\Psi = id$  and  $\Psi\Phi = id$ .

To see that  $\Phi$  is an algebra morphism, we only need to show that  $\Phi' = (id \otimes \lambda)\Phi$  is an algebra morphism. Set  $\xi = (id \otimes \rho)(S^{-1} \otimes \eta_H)C_{H^*,R}\phi$ , which is a morphism from  $R$  to  $R \otimes (H \bar{\otimes} H^{\hat{*}})$ . We have that  $\xi$  is an algebra morphism and  $\Phi' = (id \otimes m)(\xi \otimes \lambda)$ . Using Lemma 2.5, we can show that  $(id \otimes m)(C_{H \bar{\otimes} H^{\hat{*}},R} \otimes id)(\lambda \otimes \xi) = (id \otimes m)(\xi \otimes \lambda)(\alpha \otimes id \otimes id)(id \otimes C_{H,R} \otimes id)(id \otimes id \otimes C_{H^*,R})(\Delta \otimes id \otimes id)$ . Applying this, we see that  $\Phi'$  is an algebra morphism.  $\square$

### 3 Drinfeld Double

In this section, we construct the Drinfeld double  $D(H)$  for a finite Hopf algebra  $H$  with  $C_{H,H} = C_{H,H}^{-1}$  in the braided tensor category  $\mathcal{C}$ . We show that  $(D(H), [b])$  is quasi-triangular.

**Theorem 3.1** Let  $H$  and  $A$  be two bialgebras in  $\mathcal{C}$ . Assume that  $\tau$  is an invertible skew pairing on  $H \otimes A$  and symmetric with respect to the braiding. If we define  $\alpha = (\tau \otimes id \otimes \bar{\tau})(id \otimes id \otimes C_{H,A} \otimes id)(id \otimes C_{H,A} \otimes \Delta)(\Delta \otimes \Delta)$  and  $\beta = (\tau \otimes id \otimes \bar{\tau})(id \otimes C_{H,A} \otimes id \otimes id)(\Delta \otimes C_{H,A} \otimes id)(\Delta \otimes \Delta)$ , then the double cross product  $A_\alpha \bowtie_\beta H$ , defined in [16, P36], of  $A$  and  $H$  is an algebra and a coalgebra. If  $A$  and  $H$  are Hopf algebras, then  $A_\alpha \bowtie_\beta H$  has an antipode. Furthermore,  $A_\alpha \bowtie_\beta H$  is a bialgebra if and only if  $C_{A,H}C_{H,A} = id$ .

**Proof.** We can check that  $(A, \alpha)$  is an  $H$ -module coalgebra and  $(H, \beta)$  is an  $A$ -module coalgebra step by step. We can also check that (M1)–(M3) in [16, P36–37] hold step by step. Consequently, it follows from [16, Corollary 1.8, Theorem 1.5] that  $A_\alpha \bowtie_\beta H$  is an algebra and a coalgebra. Observing the proof of [16, Theorem 1.5], we know that the condition (M4) is not needed in the proof. Consequently,  $A_\alpha \bowtie_\beta H$  has an antipode.

From [1, Proposition 3.6], we obtain our last assertion.  $\square$

In this case,  $A_\alpha \bowtie_\beta H$  can be written as  $A \bowtie_\tau H$  and called a double cross product.

**Theorem 3.2** *Let  $H$  be a finite Hopf algebra with  $C_{H,H} = C_{H,H}^{-1}$ . Set  $A = (H^*)^{op}$  and  $\tau = d_H C_{H,A}$ . Then  $(D(H), [b])$  is a quasi-triangular Hopf algebra in  $\mathcal{C}$  with  $[b] = \eta_A \otimes b \otimes \eta_H$  and  $D(H) = A \bowtie_\tau H$ , called the Drinfeld double of  $H$ .*

**Proof.** Using [8, Proposition 2.4] or the definition of the evaluation and coevaluation on tensor product, we can obtain that  $\tau$  is a skew pairing on  $H \otimes A$  and  $[b]$  satisfies (QT1) and (QT2). For (QT3), see that  $(m \otimes m)(id \otimes C_{D(H), D(H)} \otimes id)(\Delta^{cop} \otimes [b]) = (id \otimes id \otimes m \otimes id)(id \otimes C_{A,H} \otimes id \otimes id)(id \otimes id \otimes m \times id \otimes id)(id \otimes id \otimes id \otimes m \times id \otimes id)(id \otimes id \otimes id \otimes m \times id \otimes id)(C_{A,A} \otimes id \otimes id \otimes C_{H,H} \times C_{H,A})(id \otimes id \otimes id \otimes C_{H,H} \otimes C_{H,H} \otimes id)(id \otimes id \otimes C_{H,H} \otimes C_{H,H} \otimes b)(id \otimes id \otimes id \otimes C_{H,H} \otimes S^{-1})(\Delta \otimes \Delta \otimes C_{H,H})(id \otimes \Delta \otimes id)(id \otimes \Delta) = (id \otimes C_{A,H} \otimes id)(id \otimes m \otimes id \otimes id)(C_{A,A} \otimes C_{H,A} \otimes id)(id \otimes id \otimes m \otimes id \otimes id)(id \otimes id \otimes C_{H,H} \otimes C_{H,A})(id \otimes id \otimes id \otimes C_{H,H} \otimes id)(\Delta \otimes \Delta \otimes b)$  and  $(m \otimes m)(id \otimes C_{D(H), D(H)} \otimes id)([b] \otimes \Delta) = (id \otimes m \otimes id \otimes id)(C_{H,A} \otimes C_{A,H} \otimes id)(id \otimes id \otimes m \otimes id \otimes id)(id \otimes id \otimes m \otimes id \otimes id \otimes id)(id \otimes id \otimes m \otimes id \otimes id \otimes id)(id \otimes C_{A,A} \otimes id \otimes S \otimes id \otimes id \otimes id)(id \otimes id \otimes C_{A,A} \otimes id \otimes id \otimes id \otimes id)(id \otimes C_{A,A} \otimes \Delta \otimes id \otimes id \otimes id)(b \otimes \Delta \otimes id \otimes id \otimes id)(\Delta \otimes \Delta) = (id \otimes C_{A,H} \otimes id)(id \otimes m \otimes id \otimes id)(C_{A,A} \otimes C_{H,A} \otimes id)(id \otimes id \otimes m \otimes id \otimes id)(id \otimes id \otimes C_{H,H} \otimes C_{H,A})(id \otimes id \otimes m \otimes id \otimes id)(id \otimes id \otimes C_{H,H} \otimes id)(\Delta \otimes \Delta \otimes b)$ . Thus (QT3) holds.  $\square$

In fact, there exists a very closed relation between the Drinfeld double  $D(H)$  defined in Theorem 3.2 and the Drinfeld double  $\mathcal{D}(H)$  defined in [3]. Since  $C_{H^*, H} : D(H) = H^{*op} \bowtie_\tau H \longrightarrow \mathcal{D}(H^{op}) = H^{op} \bowtie_d H^*$  is an anti-algebra isomorphism and a coalgebra isomorphism, we have  $D(H) \cong \mathcal{D}(H^{op})^{op}$  as Hopf algebras in  $\mathcal{C}$ .

## 4 Example

In this section, using preceding conclusions, we give some examples for the duality theorem and Drinfeld double in a braided tensor category  $\mathcal{C}$ .

**Proposition 4.1** *(See [6, Definition 2.8 (R2)]) A  $\chi$ -Hopf algebra  $H$  is a Hopf algebra living in the braided tensor category  $({}^k\mathcal{M}, C^r)$ , where  $r(g, h) = v^{\chi(g, h)}$  for any  $g, h \in G$ ;  $\chi$  is a map from  $G \times G$  to  $\mathbf{Z}$  with  $\chi(a, b) = \chi(b, a)$  and  $\chi(a + b, c) = \chi(a, c) + \chi(b, c)$  for any  $a, b, c \in G$ .*

It follows from the preceding proposition and [6, Ex. 2.12, Section 3] that Lusztig's algebra  $'f$ , the twisted Ringel-Hall algebra and Ringel's composition algebra are all Hopf algebras in braided tensor categories.

**Example 4.2** *The bilinear map  $\tau$ , defined in [5, Pro. 1.2.3], of Lusztig's algebra  $'f$  is symmetric with respect to the braiding.*

**Proof.** For any homogeneous elements  $x, y, z \in 'f$ , we have that  $(\tau \otimes id)(id \otimes C_{'f, 'f})(x \otimes y \otimes z) = \tau(x, z)v^{|y| \cdot |z|} \delta_{|x|, |z|} = \tau(x, z)v^{|y| \cdot |x|} \delta_{|x|, |z|} = (id \otimes \tau)(C_{'f, 'f} \otimes id)(x \otimes y \otimes z)$ .  $\square$

**Example 4.3** (see [9, Example 9.4.9]) The evaluation of the braided group analogue  $\underline{H}$  of an ordinary coquasi-triangular cocommutative Hopf algebra  $(H, r)$  is symmetric with respect to braiding  $C^r$ . In particular, the above conclusion holds for  $H = kG$ .

**Proof.** It is straightforward.  $\square$

**Example 4.4** Let  $H$  denote Lusztig's algebra  $'f$ . If  $A = 'f^{op}$ , then the bilinear map  $\tau$  as in Example 4.2 is a skew pairing on  $H \otimes A$  and symmetric with respect to the braiding. Thus, by Theorem 3.1,  $A \bowtie_\tau H$  is an algebra and a coalgebra with an antipode in  $({}^{k\mathbf{Z}[I]}\mathcal{M}, C^r)$ , but it is never a bialgebra in  $({}^{k\mathbf{Z}[I]}\mathcal{M}, C^r)$  since  $C_{A,H}C_{H,A} \neq id$ .

It has been known that the category of comodules of every ordinary coquasi-triangular Hopf algebra is a braided tensor category. For example, let  $H = \mathbf{C}\mathbf{Z}_n$  and  $r(a, b) = (e^{\frac{2\pi i}{n}})^{ab}$  for any  $a, b \in \mathbf{Z}_n$ , where  $\mathbf{C}$  is the complex field. It is clear that  $(\mathbf{C}\mathbf{Z}_n, r)$  is a coquasi-triangular Hopf algebra. Thus,  $(\mathbf{C}\mathbf{Z}_n\mathcal{M}, C^r)$  is a braided tensor category, usually written as  $\mathcal{C}_n$ . Every algebra or Hopf algebra living in  $\mathcal{C}_n$  is called an anyonic algebra or anyonic Hopf algebra (see [9, Example 9.2.4]). Every algebra or Hopf algebra living in  $\mathcal{C}_2$  is called a superalgebra or super-Hopf algebra. In particular,  $\mathcal{C}_n$  is a strictly braided tensor category when  $n > 2$ .

It follows from Theorem 3.2 that

**Corollary 4.5** (Duality Theorem) Let  $H$  be a finite dimensional Hopf algebra with  $C_{H,H} = C_{H,H}^{-1}$  in  $(\mathcal{C}, C)$ . Then

$$(R\#H)\#H^{\hat{*}} \cong M_n(R) \quad \text{as algebras in } (\mathcal{C}, C)$$

in the following three cases:

- (i)  $(\mathcal{C}, C)$  is the braided tensor category  $({}_B\mathcal{M}, C^R)$  determined by the quasi-triangular structure  $R$ ;
- (ii)  $(\mathcal{C}, C)$  is the braided tensor category  $({}^B\mathcal{M}, C^r)$  determined by the coquasi-triangular structure  $r$
- (iii)  $(\mathcal{C}, C)$  is the braided tensor category  ${}_B^B\mathcal{YD}$  or  ${}_B\mathcal{YD}^B$  of Yetter-Drinfeld modules.

**Example 4.6** (see [10]) Let  $A$  denote the anyonic line algebra, i.e.,  $A = \mathbf{C}\{x\}/\langle x^n \rangle$ , where  $\mathbf{C}\{x\}$  is a free algebra over the complex field  $\mathbf{C}$  and  $\langle x^n \rangle$  is an ideal generalized by  $x^n$  of  $\mathbf{C}\{x\}$ . Set  $\mathbf{C}[\xi] = \mathbf{C}\{x\}/\langle x^n \rangle$  with  $\xi^n = 0$ . Its comultiplication, counit, and antipode are

$$\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi, \quad \epsilon(\xi) = 0, \quad \text{and} \quad S(\xi) = -\xi,$$

respectively. It is straightforward to check that  $H$  is an anyonic Hopf algebra. Let  $H$  denote the braided group analogue  $\underline{\mathbf{C}\mathbf{Z}_n}$  of  $\mathbf{C}\mathbf{Z}_n$ . Since  $\mathbf{C}\mathbf{Z}_n$  is commutative, we have  $C_{\mathbf{C}\mathbf{Z}_n, \mathbf{C}\mathbf{Z}_n} = C_{\mathbf{C}\mathbf{Z}_n, \mathbf{C}\mathbf{Z}_n}^{-1}$ . Let  $H$  act on  $A$  trivially. By Corollary 4.5, we have

$$(\mathbf{C}[\xi]\#\underline{\mathbf{C}\mathbf{Z}_n})\#(\underline{\mathbf{C}\mathbf{Z}_n})^{\hat{*}} \cong M_n(\mathbf{C}[\xi]) \quad \text{as algebras in } (\mathbf{C}\mathbf{Z}_n\mathcal{M}, C^r).$$

By the way, we also have that the Drinfeld double of  $\underline{\mathbf{C}\mathbf{Z}_n}$  is quasi-triangular in  $(\mathbf{C}\mathbf{Z}_n\mathcal{M}, C^r)$  with the quasi-triangular structure  $[b]$ .

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## References

- [1] Y. Bespalov and B.Drabant, Cross product bialgebras, I, J. algebra, **219** (1999), 466–505.
- [2] R. J. Blattner and S. Montgomery, A duality theorem for Hopf module algebras, J. algebra, **95** (1985), 153–172.
- [3] H.X.Chen, Quantum double in monoidal categories, Comm.Algebra, **28** (2000)5, 2303–2328.
- [4] V. G. Drinfeld, Quantum groups, in “Proceedings International Congress of Mathematicians, August 3-11, 1986, Berkeley, CA” pp. 798–820, Amer. Math. Soc., Providence, RI, 1987.
- [5] G.Lusztig, Introduction to Quantum Groups, Progress on Math., Vol.110, Birkhauser, Berlin, 1993.
- [6] L.Li and P.Zhang, Twisted Hopf algebras, Ringel-Hall algebras and Green’s categories, J.Algebra, **231**(2000), P713–743.
- [7] S. Majid, Physics for algebraists: Non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction, J.Algebra **130** (1990), 17–64.
- [8] S. Majid, Algebras and Hopf algebras in braided categories, Lecture notes in pure and applied mathematics advances in Hopf algebras, Vol. 158, edited by J. Bergen and S. Montgomery, Marcel Dekker, New York, 1994, 55–105.
- [9] S. Majid, Foundations of Quantum Group Theory, Cambridge University Press, Cambridge, 1995.
- [10] S. Majid and M.J. Rodriguez Piazza, Random walk and the heat equation on superspace and anyspace, J. Math. Phys., **35**(1994) 7, 3753–3760.
- [11] S. Montgomery, Hopf Algebras and Their Actions on Rings. CBMS Number 82, AMS, Providence, RI, 1993.
- [12] D. E. Radford, Minimal quasi-triangular Hopf algebras, J. algebra, **157** (1993), 281–315.
- [13] S. Stratila, Modular Theory in Operator Algebras, Editura Academiei and Abacus Press, Bucuresti, 1981.
- [14] M.E.Sweedler, Hopf Algebras, Benjamin, New York, 1969.

- [15] M. Takeuchi, Finite Hopf algebras in braided tensor categories , J.Pure and Applied Algebra, **138**(1999), 59-82.
- [16] Shouchuan Zhang, Hui-Xiang Chen, The double bicrossproducts in braided tensor categories, Communications in Algebra, **29**(2001)1, P31–66.



We denote the multiplication, comultiplication, antipode, braiding and inverse braiding by

The proof of Lemma2.4  
Proof.



The diagram illustrates the proof that  $\lambda$  is an algebraic morphism. It consists of several stages of string diagrams connected by equals signs:

- Stage 1:** Two copies of  $H \# H^*$  are connected by a node labeled  $\lambda$ . This node is connected to  $H \bar{\otimes} H^*$ .
- Stage 2:** The diagram is transformed into a more complex structure involving  $H$  and  $H^*$  strands with crossings and loops.
- Stage 3:** Further transformations lead to a diagram with multiple crossings and loops, involving  $H$  and  $H^*$  strands.
- Stage 4:** The diagram is simplified to a structure involving  $H$  and  $H^*$  strands with crossings and loops.
- Stage 5:** The final stage shows the diagram simplifying back to the initial structure, where two copies of  $H \# H^*$  are connected by a  $\lambda$  node, which is then connected to  $H \bar{\otimes} H^*$ .

Thus  $\lambda$  is an algebraic morphism.

Similarly, we can show that  $\rho$  is an anti-algebraic morphism.  $\square$

The proof of Theorem 2.5.

$$\begin{array}{c}
\begin{array}{c} H \# H^{\hat{*}} \quad H^{\hat{*}} \# H \\ \downarrow \quad \downarrow \\ \textcircled{\lambda} \quad \textcircled{\rho} \\ \downarrow \\ H \bar{\otimes} H^{\hat{*}} \end{array} \\
= \\
\begin{array}{c}
\begin{array}{cccc} H & H^{\hat{*}} & H^{\hat{*}} & H \end{array} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\begin{array}{c} \textcircled{S} \quad \textcircled{\neg} \quad \textcircled{\neg} \quad \textcircled{\lambda} \quad \textcircled{\rho} \end{array} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\begin{array}{c} \textcircled{\rho} \quad \textcircled{\lambda} \end{array} \\
\downarrow \\
H \bar{\otimes} H^{\hat{*}}
\end{array}
\end{array} \quad \dots (1)$$

We show (1) by following five steps. It is easy to check the following (i) and (ii).

(i)

$$\begin{array}{c}
\begin{array}{c} H \# \eta_{H^{\hat{*}}} \quad \eta_{H^{\hat{*}}} \# H \\ \downarrow \quad \downarrow \\ \textcircled{\lambda} \quad \textcircled{\rho} \\ \downarrow \\ H \bar{\otimes} H^{\hat{*}} \end{array} \\
= \\
\begin{array}{c}
\begin{array}{cc} H \# \eta_{H^{\hat{*}}} & \eta_{H^{\hat{*}}} \# H \end{array} \\
\downarrow \quad \downarrow \\
\begin{array}{c} \textcircled{\rho} \quad \textcircled{\lambda} \end{array} \\
\downarrow \\
H \bar{\otimes} H^{\hat{*}}
\end{array}
\end{array}$$

(ii)

$$\begin{array}{c}
\begin{array}{c} \eta_H \# H^{\hat{*}} \quad H^{\hat{*}} \# \eta_H \\ \downarrow \quad \downarrow \\ \textcircled{\lambda} \quad \textcircled{\rho} \\ \downarrow \\ H \bar{\otimes} H^{\hat{*}} \end{array} \\
= \\
\begin{array}{c}
\begin{array}{cc} \eta_H \# H^{\hat{*}} & H^{\hat{*}} \# \eta_H \end{array} \\
\downarrow \quad \downarrow \\
\begin{array}{c} \textcircled{\rho} \quad \textcircled{\lambda} \end{array} \\
\downarrow \\
H \bar{\otimes} H^{\hat{*}}
\end{array}
\end{array}$$

$$(\ddot{\text{iii}})$$

$$\begin{array}{c}
 \begin{array}{c}
 H^{\hat{*}} \# \eta_H \quad H^{\hat{*}} \# \eta_H \\
 \begin{array}{c} \circ \rho \quad \circ \lambda \end{array} \\
 \downarrow \\
 H \bar{\otimes} H^{\hat{*}}
 \end{array}
 =
 \begin{array}{c}
 \begin{array}{cccc}
 H^{\hat{*}} & \eta_H & H & \eta_{H^{\hat{*}}}
 \end{array} \\
 \begin{array}{c}
 \text{[Diagram with crossings and nodes } \circ \lambda, \circ \rho \text{]} \\
 \downarrow \\
 H \bar{\otimes} H^{\hat{*}}
 \end{array}
 \end{array}
 .
 \end{array}$$

In fact, the right side
 
$$\begin{array}{c}
 \begin{array}{cccc}
 H^{\hat{*}} & \eta_H & H & \eta_{H^{\hat{*}}}
 \end{array} \\
 \begin{array}{c}
 \text{[Diagram with crossings and nodes } \circ \lambda, \circ \rho \text{]} \\
 \downarrow \\
 H \bar{\otimes} H^{\hat{*}}
 \end{array}
 \end{array}
 =$$

$$\begin{array}{c}
 \begin{array}{cc}
 H^{\hat{*}} & H \\
 \downarrow & \downarrow \\
 \begin{array}{c}
 \text{[Diagram with crossings and nodes } \circ \lambda, \circ \rho \text{]} \\
 \downarrow \\
 H \bar{\otimes} H^{\hat{*}}
 \end{array}
 \end{array}
 = \text{the left side.}
 \end{array}$$

Thus (iii) holds.

(iv)

$$\begin{array}{c}
 \begin{array}{c}
 H \# \eta_{H^*} \quad H^* \# \eta_H \\
 \downarrow \quad \downarrow \\
 \textcircled{\lambda} \quad \textcircled{\rho} \\
 \downarrow \\
 H \bar{\otimes} H^*
 \end{array}
 =
 \begin{array}{c}
 \begin{array}{c}
 H^* \quad \eta_{H^*} \quad H^* \quad H \\
 \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
 \textcircled{S} \quad \textcircled{\rho} \quad \textcircled{-} \quad \textcircled{\lambda} \\
 \downarrow \\
 H \bar{\otimes} H^*
 \end{array}
 \end{array}
 .
 \end{array}$$
  

$$\begin{array}{c}
 \begin{array}{c}
 H^* \quad \eta_{H^*} \quad H^* \quad \eta_H \\
 \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
 \textcircled{S} \quad \textcircled{-} \quad \textcircled{\rho} \quad \textcircled{\lambda} \\
 \downarrow \\
 H \bar{\otimes} H^*
 \end{array}
 \stackrel{\text{The right side by (iii)}}{=}
 \begin{array}{c}
 \begin{array}{c}
 H^* \quad \eta_{H^*} \quad H^* \quad \eta_H \\
 \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
 \textcircled{S} \quad \textcircled{-} \quad \textcircled{\rho} \quad \textcircled{\lambda} \\
 \downarrow \\
 H \quad H^*
 \end{array}
 \end{array}$$

$$= \text{[Diagram with strands } H, H^*, H, H^* \text{ and crossings]} = \text{[Simplified diagram with strands } H, H^*]$$

and

$$\text{the left side} = \text{[Diagram with strands } H, \eta_{H^*}, H^*, \eta_H \text{ and crossings]} = \text{[Simplified diagram with strands } H, H^*]$$

Thus (iv) holds.

(v)

$$\begin{array}{c} \eta_H \# H^* \\ \downarrow \lambda \\ \eta_{H^*} \# H \\ \downarrow \rho \\ H \otimes H^* \end{array} = \text{[Complex diagram with strands } \eta_H, H^*, \eta_{H^*}, H \text{ and crossings]} .$$

In fact, the right side =

and

the left side =

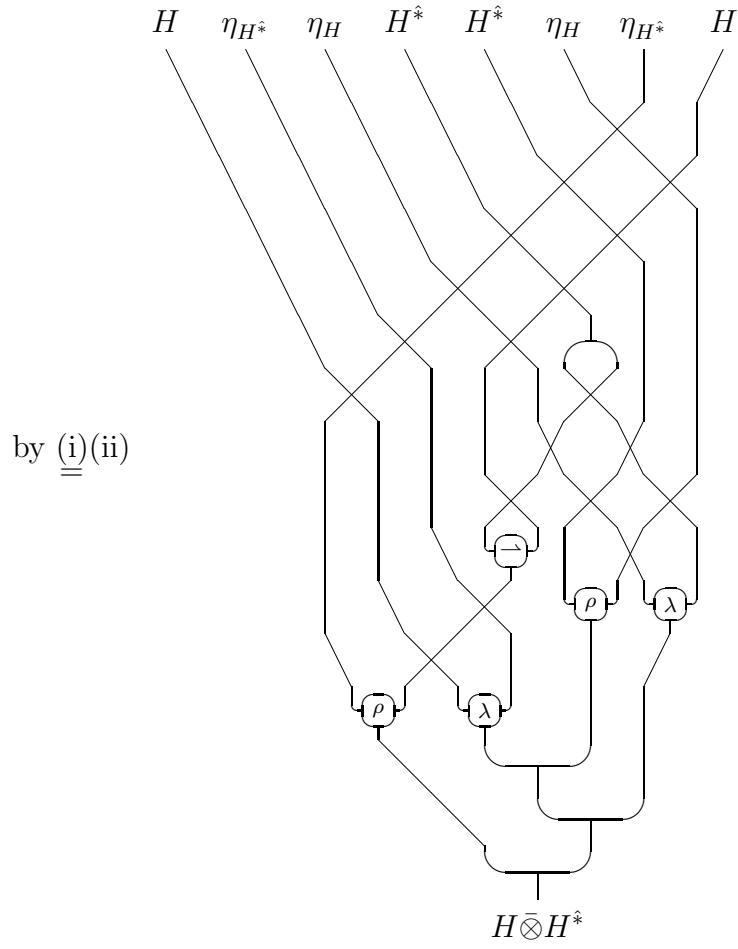
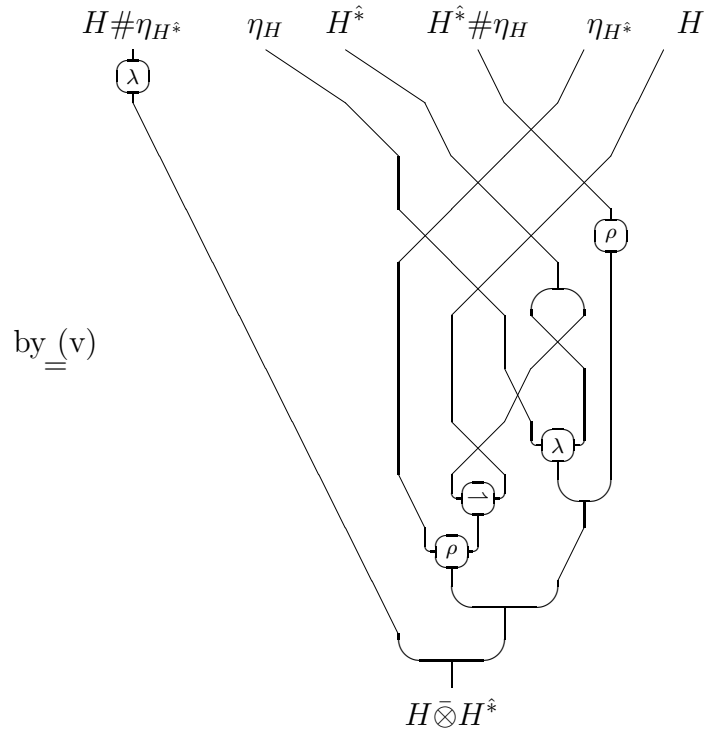
$$\begin{array}{c}
\begin{array}{c} H^* \quad H \\ \text{[Diagram 1: A complex braid-like structure with four strands labeled } H^* \text{ and } H \text{ at the top and bottom. It features several crossings and loops, including a central box-like structure.]}\end{array} \\
= \\
\begin{array}{c} H^* \quad H \\ \text{[Diagram 2: A second complex braid-like structure, similar to Diagram 1 but with different strand connections.]}\end{array} \\
= \text{ the right side.}
\end{array}$$

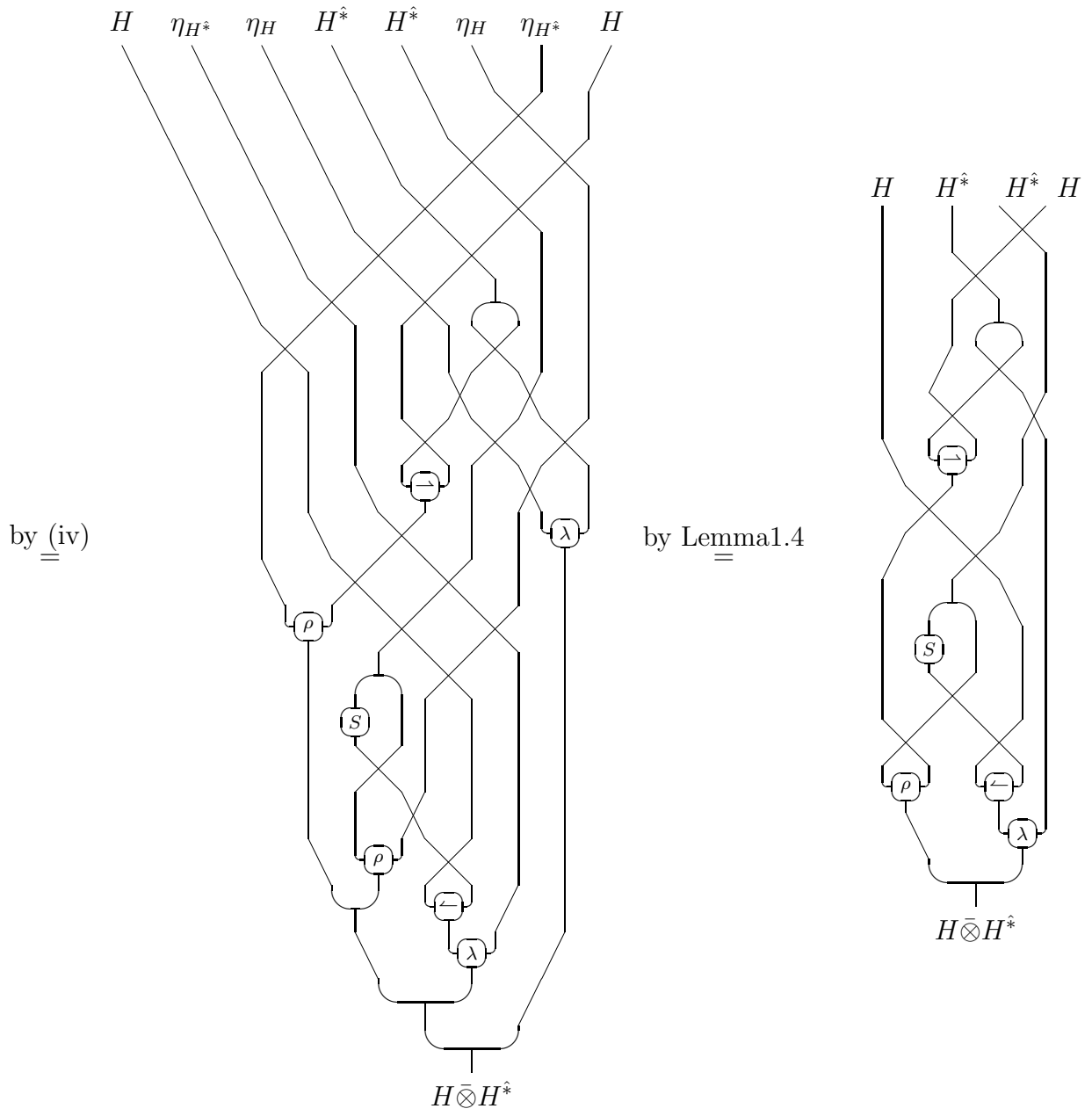
Thus (v) holds.

Now we show that the relation (1) holds.

$$\begin{array}{c}
\begin{array}{c} H \# \eta_{H^*} \quad \eta_H \# H^* \quad H^* \# \eta_H \quad \eta_{H^*} \# H \\ \text{[Diagram 3: A diagram showing four inputs at the top, each passing through a box labeled } \lambda \text{. The first two } \lambda \text{ boxes are connected by a line to a bracketed node. The last two } \lambda \text{ boxes are connected by a line to another bracketed node. These two nodes are then connected by a line to a final bracketed node labeled } H \bar{\otimes} H^* \text{ at the bottom.}]\end{array} \\
\text{the left side of (1)} \stackrel{\text{by Lemma 1.4}}{=} \text{[Diagram 3]}
\end{array}$$







$$\begin{array}{c}
H \quad H^{\hat{*}} \quad H^{\hat{*}} \quad H \\
\text{[Diagram: A complex braiding of four strands with nodes } S, \rho, \lambda \text{ and crossings]} \\
= \text{ [Simplified diagram with nodes } \rho, \lambda \text{]} = \text{ the right side of (1).} \quad \square \\
H \bar{\otimes} H^{\hat{*}}
\end{array}$$

The proof of Theorem 2.8 .

(i)

$$\text{Let } \begin{array}{c} H^{\hat{*}} \\ | \\ \textcircled{w} \\ | \\ H \# H^{\hat{*}} \end{array} = \begin{array}{c} H^{\hat{*}} \\ | \\ \textcircled{S} \\ | \\ \textcircled{\rho} \\ | \\ \textcircled{\lambda^{-1}} \\ | \\ H \# H^{\hat{*}} \end{array} \quad \text{and} \quad \begin{array}{c} R \\ | \\ \textcircled{\phi} \\ | \\ H^{\hat{*}} \quad R \end{array} = \begin{array}{c} R \\ | \\ \textcircled{b'} \\ | \\ \textcircled{\alpha} \\ | \\ H^{\hat{*}} \quad R \end{array} = \begin{array}{c} R \\ | \\ \text{[Crossing]} \\ | \\ \textcircled{\alpha} \\ | \\ H^{\hat{*}} \quad R \end{array}, \text{ where } b' = C_{H, H^*} b_H.$$

We define

$$\begin{array}{c} R \otimes (H \# H^{\hat{*}}) \\ | \\ \textcircled{\Psi} \\ | \\ (R \# H) \# H^{\hat{*}} \end{array} = \begin{array}{c} R \quad H \quad H^{\hat{*}} \\ | \quad | \quad | \\ \text{[Diagram with nodes } S, \phi, w, \lambda^{-1} \text{]} \\ | \quad | \quad | \\ R \quad H \quad H^{\hat{*}} \end{array} \quad \text{and} \quad \begin{array}{c} (R \# H) \# H^{\hat{*}} \\ | \\ \textcircled{\Phi} \\ | \\ R \otimes (H \# H^{\hat{*}}) \end{array} = \begin{array}{c} R \quad H \quad H^{\hat{*}} \\ | \quad | \quad | \\ \text{[Diagram with nodes } \phi, w, \lambda \text{]} \\ | \quad | \quad | \\ R \quad H \quad H^{\hat{*}} \end{array}.$$

We see that

$$\begin{aligned}
\Psi\Phi &= \text{Diagram 1} \quad \text{since } w \text{ is algebraic} \quad \text{Diagram 2} \\
&= \text{Diagram 3} = \text{Diagram 4} = \text{Diagram 5}
\end{aligned}$$

The diagrams are string diagrams representing algebraic relations. Diagram 1 shows a complex network of nodes  $S, w, \alpha, b', R, H\#H^{\hat{*}}$ . Diagram 2 shows a simplified version where  $w$  is moved. Diagram 3 shows further simplification with loops. Diagram 4 shows the removal of  $S$  and  $w$ . Diagram 5 shows the final result as two parallel vertical lines labeled  $R$  and  $H\#H^{\hat{*}}$ .

Similarly, we have  $\Phi\Psi = id$ . Thus  $\Phi$  is invertible.

Now we show that  $\Phi$  is algebraic.

$$\text{Let } (R\#H)\#H^{\hat{*}} \xrightarrow{\Phi'} R \otimes (H \bar{\otimes} H^{\hat{*}}) = \text{Diagram 6}$$

Diagram 6 is a string diagram with nodes  $\bar{S}, \alpha, \rho, \lambda, R, \eta_H, H\#H^{\hat{*}}, H \bar{\otimes} H^{\hat{*}}$ .

It is clear that  $\Phi = (id \otimes \lambda^{-1})\Phi'$ . Consequently, we only need show that  $\Phi'$  is algebraic.

Let

$$\begin{array}{c} R \\ | \\ \xi \\ | \\ R \otimes (H \bar{\otimes} H^{\hat{*}}) \end{array} = \begin{array}{c} \begin{array}{c} R \quad \eta_H \\ | \quad | \\ \begin{array}{c} \text{---} b' \text{---} \alpha \text{---} \\ | \quad | \\ \text{---} \rho \text{---} \end{array} \\ | \quad | \\ R \quad H \bar{\otimes} H^{\hat{*}} \end{array} \end{array} .$$

We have that

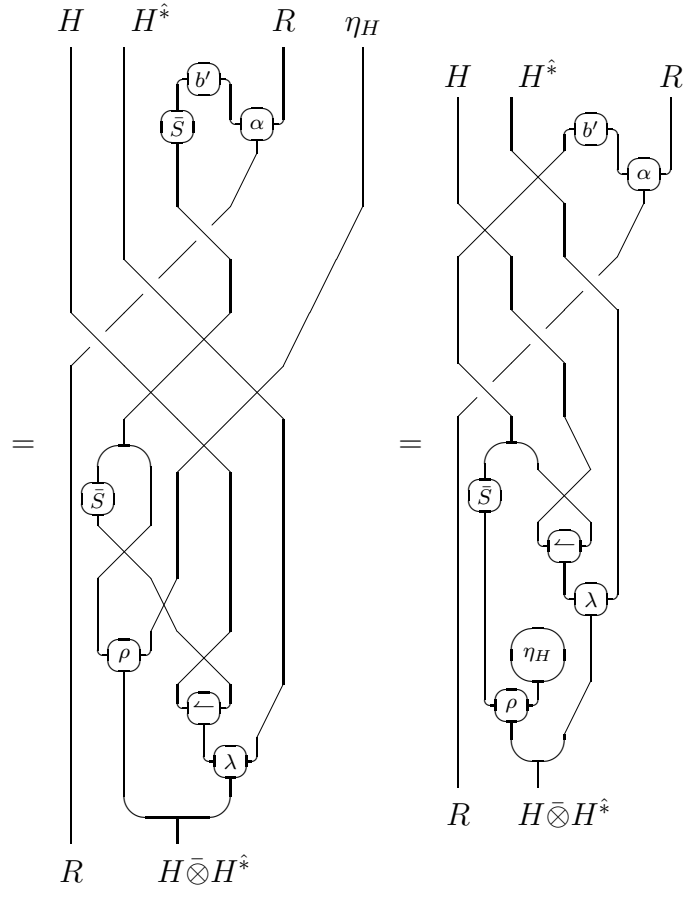
$$\begin{array}{c} (R \# H) \# H^{\hat{*}} \\ | \\ \Phi' \\ | \\ R \otimes (H \bar{\otimes} H^{\hat{*}}) \end{array} = \begin{array}{c} \begin{array}{c} R \quad H \# H^{\hat{*}} \\ | \quad | \\ \begin{array}{c} \text{---} \xi \text{---} \lambda \text{---} \\ | \quad | \\ R \quad H \bar{\otimes} H^{\hat{*}} \end{array} \end{array} \end{array} .$$

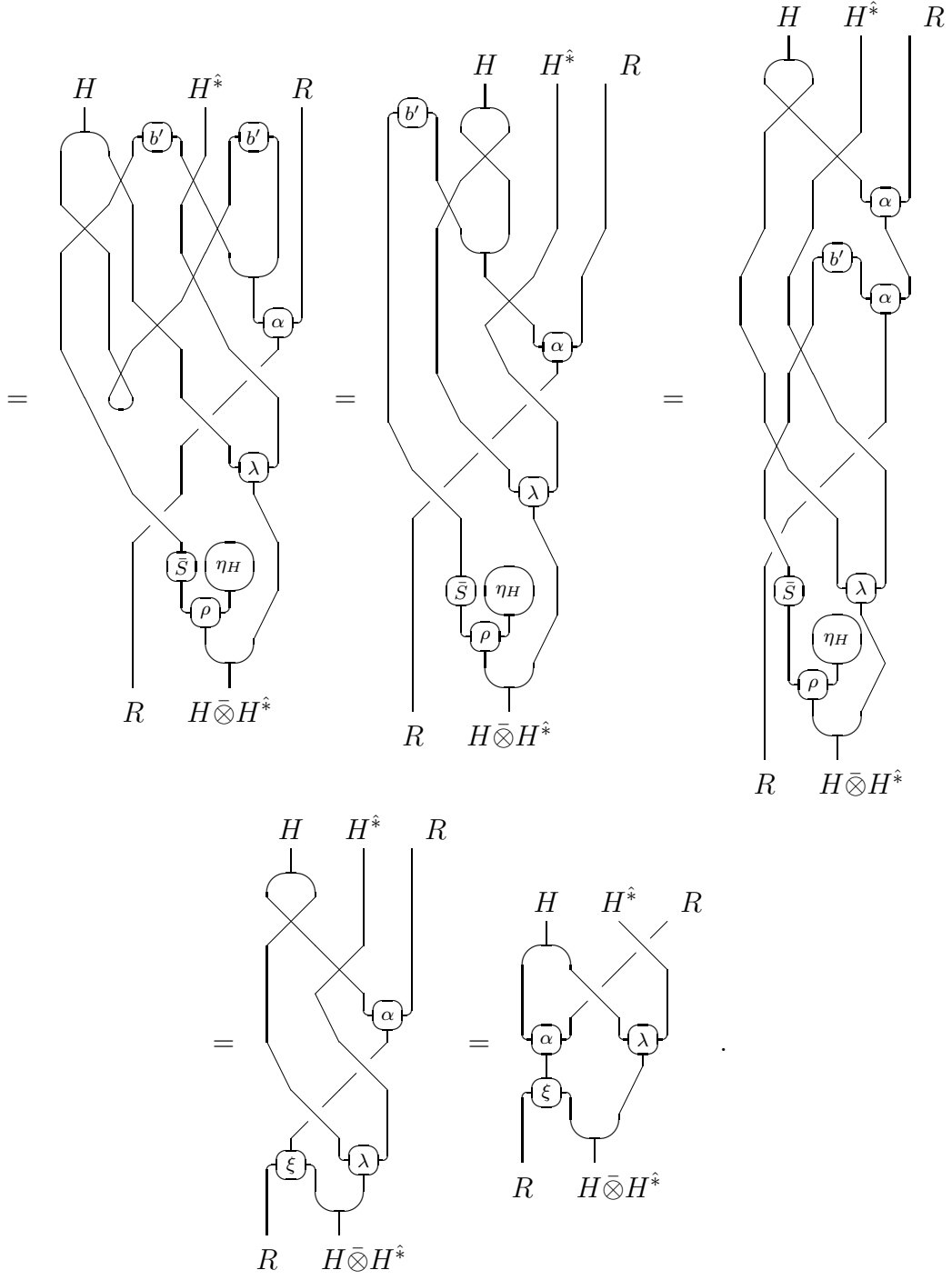
We claim that

$$\begin{array}{c} \begin{array}{c} H \quad H^{\hat{*}} \quad R \\ | \quad | \quad | \\ \begin{array}{c} \text{---} \lambda \text{---} \xi \text{---} \\ | \quad | \\ R \quad H \bar{\otimes} H^{\hat{*}} \end{array} \end{array} = \begin{array}{c} \begin{array}{c} H \quad H^{\hat{*}} \quad R \\ | \quad | \quad | \\ \begin{array}{c} \text{---} \xi \text{---} \lambda \text{---} \\ | \quad | \\ R \quad H \bar{\otimes} H^{\hat{*}} \end{array} \end{array} \end{array} . \quad \dots (*)$$

the left side =

by Lemma 1.5(1) =





Thus relation (\*) holds.



Next, we check that  $\xi$  is algebraic. We see that

$$\begin{array}{c}
\begin{array}{c} R & R \\ \downarrow \xi & \downarrow \xi \\ R \otimes (H \otimes H^*) \end{array} \\
= \\
\begin{array}{c} R & R \\ \downarrow \phi & \downarrow \phi \\ \bar{S} & \bar{S} \\ \downarrow & \downarrow \\ \eta_H & \eta_H \\ \downarrow \rho & \downarrow \rho \\ R & H \otimes H^* \end{array} \\
= \\
\begin{array}{c} R & R \\ \downarrow \phi & \downarrow \phi \\ \bar{S} & \bar{S} \\ \downarrow & \downarrow \\ \eta_H & \eta_H \\ \downarrow \rho & \downarrow \rho \\ R & H \otimes H^* \end{array} \\
= \\
\begin{array}{c} R & R \\ \downarrow \xi \\ R \otimes (H \otimes H^*) \end{array}
\end{array}$$

and obviously

$$\begin{array}{c} \eta_R \\ \downarrow \\ \xi \\ \downarrow \\ R \otimes (H \otimes H^*) \end{array}
=
\begin{array}{c} \eta_{R \otimes (H \otimes H^*)} \\ \downarrow \\ \xi \\ \downarrow \\ R \otimes (H \otimes H^*) \end{array}$$

Thus  $\xi$  is algebraic.

Now we show that  $\Phi'$  is algebraic.

$$\begin{array}{c} (R \# H) \# H^* & (R \# H) \# H^* \\ \downarrow \Phi' & \downarrow \Phi' \\ R \otimes (H \otimes H^*) \end{array}
=
\begin{array}{c} R & H & H^* & R & H & H^* \\ \downarrow \xi & \downarrow \lambda & \downarrow \xi & \downarrow \lambda \\ \downarrow & \downarrow & \downarrow & \downarrow \\ R & H \otimes H^* \end{array}$$

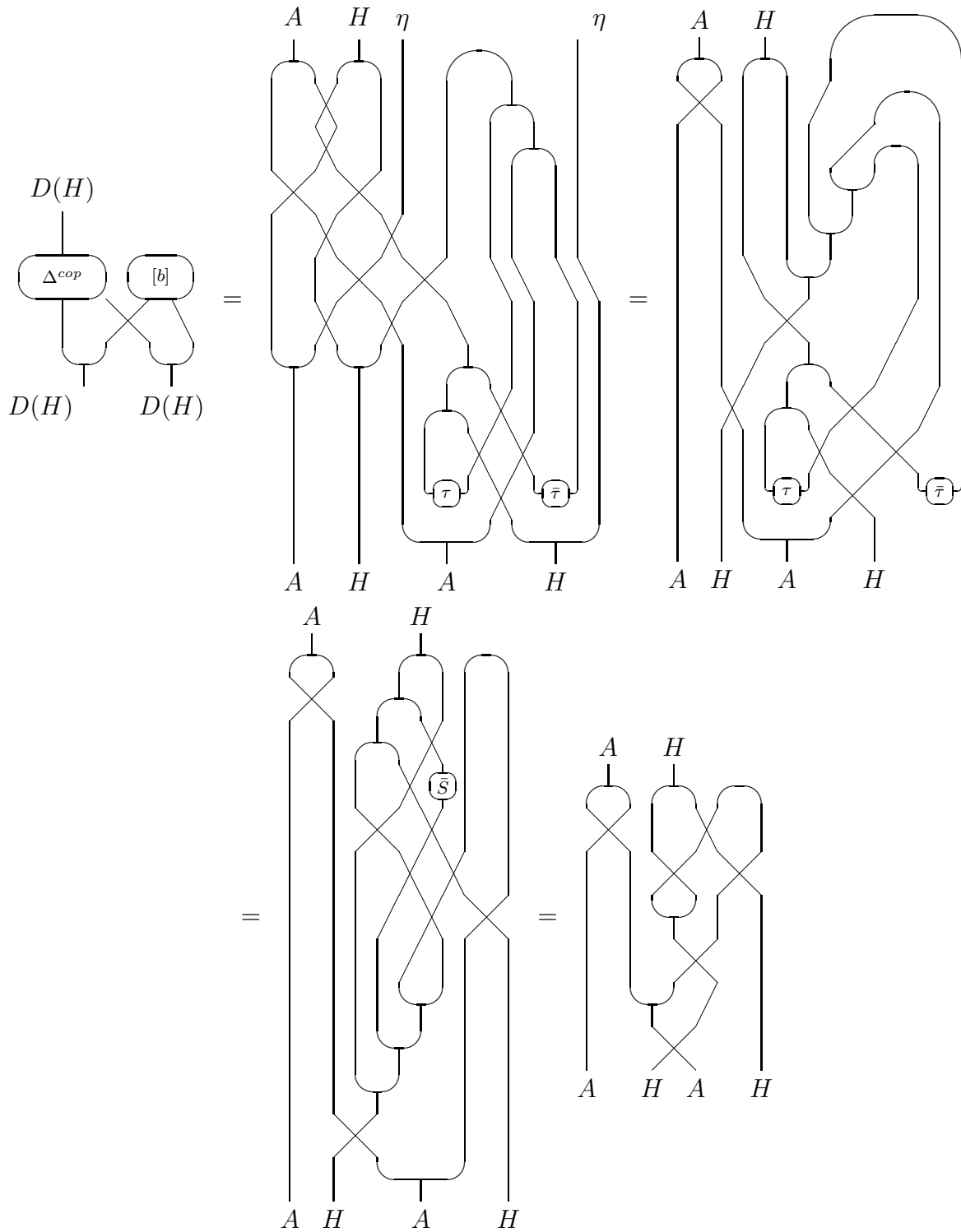
$$\begin{array}{c}
\begin{array}{c}
R \quad H \quad H^{\hat{*}} \quad R \quad H \quad H^{\hat{*}} \\
\downarrow \xi \quad \downarrow \lambda \quad \downarrow \xi \quad \downarrow \lambda \\
\text{Diagram 1}
\end{array} \\
= \\
\begin{array}{c}
R \quad H \quad H^{\hat{*}} \quad R \quad H \quad H^{\hat{*}} \\
\downarrow \xi \quad \downarrow \lambda \quad \downarrow \xi \quad \downarrow \lambda \\
\text{Diagram 2}
\end{array} \quad \text{by } (*) \\
\begin{array}{c}
R \quad H \quad H^{\hat{*}} \quad R \quad H \quad H^{\hat{*}} \\
\downarrow \xi \quad \downarrow \lambda \quad \downarrow \xi \quad \downarrow \lambda \\
\text{Diagram 3}
\end{array} \\
\text{by Lemma 1.4} \\
\begin{array}{c}
R \quad H \quad H^{\hat{*}} \quad R \quad H \quad H^{\hat{*}} \\
\downarrow \xi \quad \downarrow \lambda \quad \downarrow \xi \quad \downarrow \lambda \\
\text{Diagram 4}
\end{array} \quad \text{since } \xi \text{ is algebraic} \\
\begin{array}{c}
R \quad H \quad H^{\hat{*}} \quad R \quad H \quad H^{\hat{*}} \\
\downarrow \xi \quad \downarrow \lambda \quad \downarrow \xi \quad \downarrow \lambda \\
\text{Diagram 5}
\end{array} \\
= \\
\begin{array}{c}
(R \# H) \# H^{\hat{*}} \quad (R \# H) \# H^{\hat{*}} \\
\downarrow \\
\Phi' \\
\downarrow \\
R \otimes (H \bar{\otimes} H^{\hat{*}})
\end{array}
\end{array}$$

It is clear that

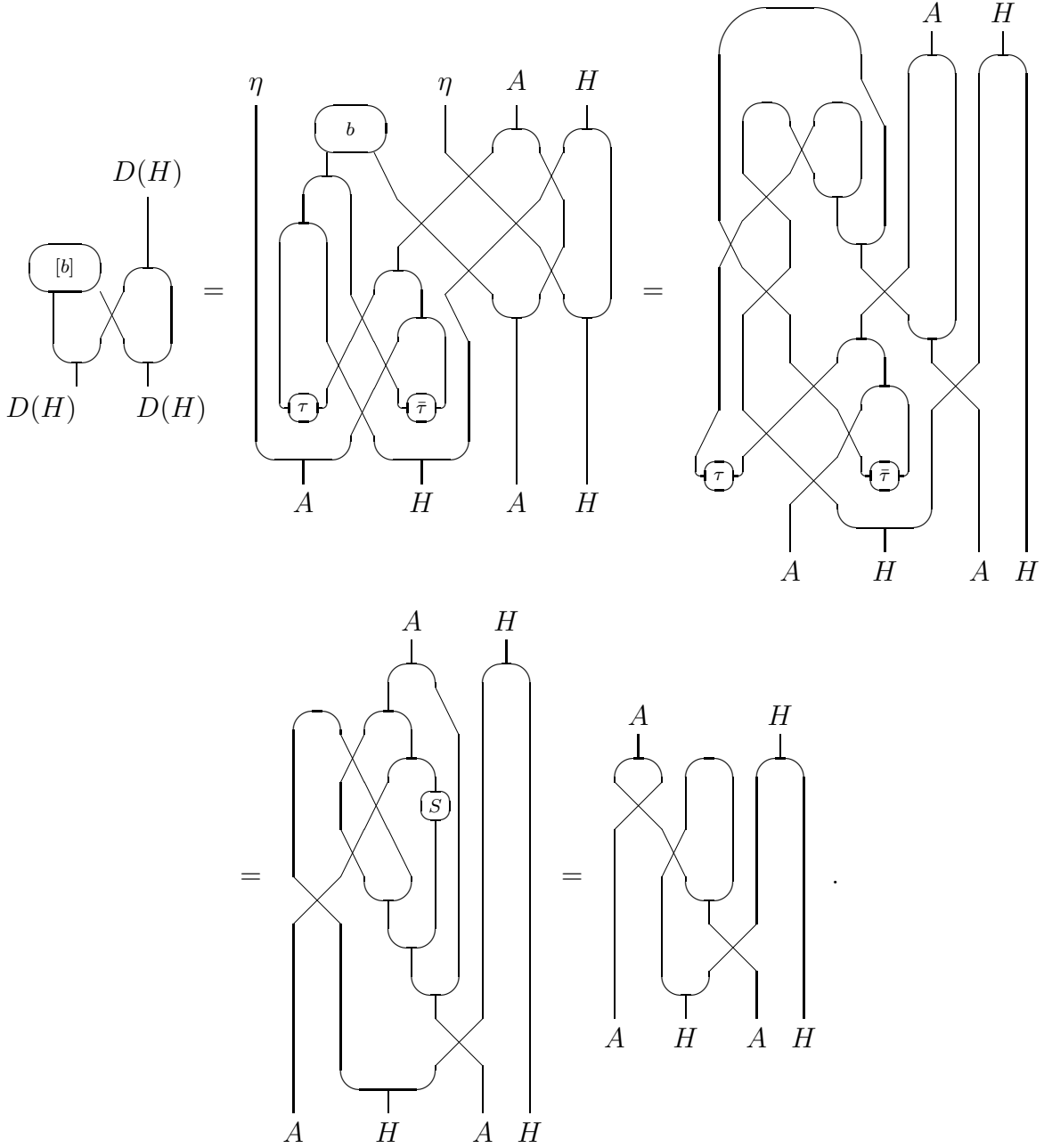
$$\begin{array}{c}
\eta_{(R \# H) \# H^{\hat{*}}} \\
\downarrow \\
\Phi' \\
\downarrow \\
R \otimes (H \bar{\otimes} H^{\hat{*}})
\end{array}
=
\begin{array}{c}
\eta_{R \otimes (H \bar{\otimes} H^{\hat{*}})} \\
\downarrow \\
R \otimes (H \bar{\otimes} H^{\hat{*}})
\end{array}$$

Thus  $\Phi'$  is algebraic.  $\square$

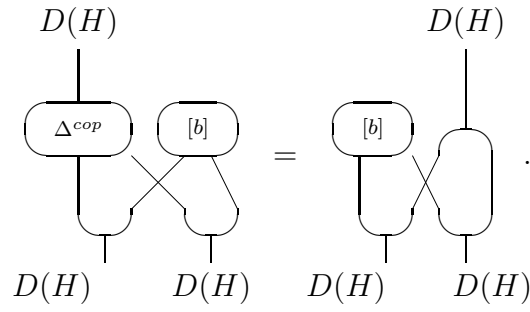
The proof of Lemma3.2.



and



Thus



We complete the proof.  $\square$

**Remark:** If  $U$  and  $V$  have left dual  $U^*$  and  $V^*$ , respectively, then  $U^* \otimes V^*$  and  $V^* \otimes U^*$  both are the left duals of  $U \otimes V$ . Their evaluations and coevaluations are

$$d_{U \otimes V} = (d_U \otimes d_V)(id_{U^*} \otimes C_{V^*, U} \otimes id_V), \quad b_{U \otimes V} = (id_U \otimes (C_{V, U^*})^{-1} \otimes id_{V^*})(b_U \otimes b_V);$$

$$d_{U \otimes V} = d_V(id_{V^*} \otimes d_U \otimes id_V), \quad b_{U \otimes V} = (id_{U^*} \otimes b_V \otimes id_U)b_U,$$

respectively. In this paper, we use the second.  $H^{\hat{*}}$  is the left dual of  $H$  under the first.